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ON SEMI-ANALYTIC FUNCTIONS OF TWO VARIABLES

BY MAXIME BÔCHER

WHEN functions of two variables are considered, it is customary either to assume that they are analytic in their two independent arguments, or, both arguments being restricted to real values, merely to impose certain conditions of continuity and differentiability upon them. An intermediate case is, however, not without its importance,* namely that in which, while the function is continuous in the two independent variables, it is analytic in only one of them. My object in the present note is to state and prove some of the fundamental theorems relating to this case. The extensions to the case of functions of more than two variables would seem to be obvious, and to present no serious difficulty.

The real variable x we restrict to a connected interval X of the x -axis. This interval may be finite or infinite, open or closed. The complex variable λ shall be restricted to a two-dimensional continuum Λ of the λ -plane, that is to a connected region every point of which is an interior point. For convenience we assume that Λ is simply connected.† The totality of points (x, λ) which are such that x lies in X and λ in Λ we speak of as the region (X, Λ) .

LEMMA 1. *If $f(x, \lambda)$ is continuous in (x, λ) and analytic in λ throughout the region (X, Λ) , then, as $\Delta\lambda$ approaches zero, the difference-quotient*

$$D = \frac{f(x, \lambda + \Delta\lambda) - f(x, \lambda)}{\Delta\lambda}$$

converges uniformly to the value $\partial f / \partial \lambda$ throughout any closed sub-region†† (X', Λ') of (X, Λ) .

Let us consider a regular closed curve C every point of which lies in Λ while no point of it lies in Λ' and which surrounds Λ' once in the positive

* For instance in the theory of linear differential or integral equations. Cf. for example, Kneser's paper in volume 58 of the *Mathematische Annalen*, in particular pages 109-116.

† This restriction is essential in Theorem II. The reader may readily satisfy himself that all the other theorems are equally true for multiply connected regions.

†† The term sub-region, as used in this paper means a region consisting of some or all points of the given region.

direction. Then by Cauchy's integral formula we have for all points of (X', Λ')

$$f(x, \lambda) = \frac{1}{2\pi i} \int \frac{f(x, t)}{t - \lambda} dt,$$

$$\frac{\partial f(x, \lambda)}{\partial \lambda} = \frac{1}{2\pi i} \int \frac{f(x, t)}{(t - \lambda)^2} dt,$$

the integral in each case being extended in the positive direction around C . Using the first of these formulæ, we find that, when $|\Delta\lambda|$ is less than the smallest distance k between the boundary of Λ' and the curve C , for all points of (X', Λ') the formula holds :

$$D = \frac{1}{2\pi i} \int \frac{f(x, t)}{(t - \lambda - \Delta\lambda)(t - \lambda)} dt.$$

We may therefore write

$$D - \frac{\partial f}{\partial \lambda} = \frac{\Delta\lambda}{2\pi i} \int \frac{f(x, t)}{(t - \lambda - \Delta\lambda)(t - \lambda)^2} dt.$$

The values of t and λ which occur in this formula satisfy the inequality

$$|t - \lambda| \geq k.$$

Let us now restrict $\Delta\lambda$ so that

$$|\Delta\lambda| < \frac{k}{2}.$$

Then

$$|t - \lambda - \Delta\lambda| \geq |t - \lambda| - |\Delta\lambda| \geq \frac{k}{2}.$$

If we denote by M the upper limit of $|f(x, \lambda)|$ in the closed region (X', C) and by l the length of the curve C , we have

$$\left| D - \frac{\partial f}{\partial \lambda} \right| \leq \frac{Ml}{\pi k^3} |\Delta\lambda|,$$

from which the uniform convergence of D towards $\frac{\partial f}{\partial \lambda}$ follows.

LEMMA 2. *If $f(x, \lambda)$ is continuous in (x, λ) throughout the region (X, Λ) and has a derivative $\partial f / \partial x$ which is also continuous throughout this region, then, as Δx approaches zero, the difference quotient*

$$\frac{f(x + \Delta x, \lambda) - f(x, \lambda)}{\Delta x}$$

converges uniformly to the value $\partial f / \partial x$, throughout any closed sub-region (X', Λ') of (X, Λ) .

If we split f into its real and pure imaginary parts :

$$f(x, \lambda) = \phi(x, \lambda) + i\psi(x, \lambda),$$

we need prove our theorem only for the two real functions ϕ and ψ , so that, since these functions satisfy precisely the same conditions as f , it will clearly be sufficient to prove our theorem on the supposition that f is real. In this case our difference quotient may be written

$$\frac{\partial f(x + \theta \Delta x, \lambda)}{\partial x} \quad (0 < \theta < 1).$$

No matter what constant ϵ we choose, we can, on account of the continuity of $\partial f / \partial x$, find a constant δ so small that, when $|\Delta x| < \delta$,

$$\left| \frac{\partial f(x + \Delta x, \lambda)}{\partial x} - \frac{\partial f(x, \lambda)}{\partial x} \right| < \epsilon$$

throughout the region (X', Λ') .* Consequently when $|\Delta x| < \delta$ our difference quotient differs from the derivative by a quantity which in absolute value is less than ϵ ; and thus our lemma is proved.

THEOREM I. *If $f(x, \lambda)$ is continuous in (x, λ) and analytic in λ throughout the region (X, Λ) , the same will be true of $\partial f / \partial \lambda$.*

We know from the elements of the theory of functions of a complex variable that $\partial f / \partial \lambda$ is analytic in λ . It remains then merely to show that at an arbitrary point (x', λ') of (X, Λ) $\partial f / \partial \lambda$ is continuous in (x, λ) . Let us choose a closed sub-region (X', Λ') so as to include the point (x', λ') and so that x' and λ' are interior points of X' and Λ' respectively, except that when x' is an end-point of X it shall also be an end-point of X' . If we restrict $\Delta \lambda$ to be in absolute value less than the least distance between the boundaries of Λ and Λ' , it is clear that the difference-quotient D of Lemma I is a continuous function of (x, λ) throughout the region (X', Λ') . Since, when $\Delta \lambda$ approaches

* We are here making use of the fact that a function of two variables continuous throughout a closed region is uniformly continuous there. This theorem, and the proof ordinarily given of it, hold in the case we are here considering where one variable is real the other complex.

zero, D approaches its limit uniformly throughout (X', Λ') , it follows that throughout this region, and consequently in (x', λ') , the limit $\partial f / \partial \lambda$ is also continuous, as was to be proved.

COROLLARY. *The functions*

$$\frac{\partial^n f}{\partial \lambda^n} \quad (n = 2, 3, \dots)$$

are continuous in (x, λ) and analytic in λ throughout the region (X, Λ) .

THEOREM II. *If $f(x, \lambda)$ is continuous in (x, λ) and analytic in λ throughout the region (X, Λ) , the same will be true of the function*

$$\Phi(x, \lambda) = \int_{\lambda_0}^{\lambda} f(x, \mu) d\mu$$

λ_0 being an arbitrary point in Λ .

That this function is analytic in λ is known from the elements of the theory of functions of a complex variable. To prove it continuous in (x, λ) at an arbitrary point (x', λ') of (X, Λ) we form the difference

$$\begin{aligned} \Phi(x' + \Delta x, \lambda' + \Delta \lambda) - \Phi(x', \lambda') &= [\Phi(x' + \Delta x, \lambda' + \Delta \lambda) - \Phi(x', \lambda' + \Delta \lambda)] \\ &\quad + [\Phi(x', \lambda' + \Delta \lambda) - \Phi(x', \lambda')]. \end{aligned}$$

Since Φ is an analytic, and therefore continuous, function of λ at the point (x', λ') , no matter how small the positive constant ϵ may be, a constant δ_1 can be so chosen that when $|\Delta \lambda| < \delta_1$

$$|\Phi(x', \lambda' + \Delta \lambda) - \Phi(x', \lambda')| < \frac{\epsilon}{2}.$$

Let us denote by l the length of a regular curve C lying in Λ and connecting λ_0 with λ' . About λ' as centre construct a circle which lies wholly in Λ and whose radius δ_2 is less than l . Then when $|\Delta \lambda| < \delta_2$, we may use as the path of integration from λ_0 to $\lambda' + \Delta \lambda$ the curve C and a straight line from λ' to $\lambda' + \Delta \lambda$. The length of this path is less than $2l$.

We have

$$\begin{aligned} \Phi(x' + \Delta x, \lambda' + \Delta \lambda) - \Phi(x', \lambda' + \Delta \lambda) \\ = \int_{\lambda_0}^{\lambda' + \Delta \lambda} [f(x' + \Delta x, \mu) - f(x', \mu)] d\mu. \end{aligned}$$

Consider now the closed region T which consists of the circle of radius δ_2 about λ' and so much of the curve C as does not already lie in this circle.

Consider also any closed sub-interval X' of X of which x' is an interior point, or, when x' is an end-point of X , of which x' is an end-point. Since f is continuous, and therefore uniformly continuous, in (X', T) , a constant δ_3 can be found such that when $|\Delta x| < \delta_3$ and $x' + \Delta x$ lies in X

$$|f(x' + \Delta x, \mu) - f(x', \mu)| < \frac{\epsilon}{4l}$$

throughout the region (X', T) . Consequently, since our path of integration lies in T and is of length less than $2l$, it follows that when $|\Delta \lambda| < \delta_2$, $|\Delta x| < \delta_3$ and $x' + \Delta x$ lies in X

$$|\Phi(x' + \Delta x, \lambda' + \Delta \lambda) - \Phi(x', \lambda' + \Delta \lambda)| < \frac{\epsilon}{2}.$$

Combining this with the similar inequality found above, and denoting by δ the smallest of the three quantities δ_1 , δ_2 , δ_3 , we see that when $|\Delta x| < \delta$, $|\Delta \lambda| < \delta$ and $x' + \Delta x$ lies in X

$$|\Phi(x' + \Delta x, \lambda' + \Delta \lambda) - \Phi(x', \lambda')| < \epsilon,$$

and the continuity of Φ is established.

THEOREM III. *If $f(x, \lambda)$ is continuous in (x, λ) and analytic in λ throughout the region (X, Λ) and has a derivative $\partial f / \partial x$ which is continuous in (x, λ) throughout this region, then this derivative is also analytic in λ throughout this region.*

Consider any regular closed curve C in Λ . If x and $x + \Delta x$ both lie in X , we have by Cauchy's integral theorem

$$\int \frac{f(x + \Delta x, \lambda) - f(x, \lambda)}{\Delta x} d\lambda = 0,$$

the integral being extended around C . A reference to Lemma 2 shows that, as Δx approaches zero, the integrand here converges uniformly to $\partial f / \partial x$ for all values of λ on C . Consequently

$$\int \frac{\partial f}{\partial x} d\lambda = 0,$$

the path of integration again being C . But since C is any closed regular curve in Λ , it follows from Morera's Theorem* that $\partial f / \partial x$ is analytic in λ .

* Cf. Osgood, *Funktionentheorie*, Bd. I, p. 256.

THEOREM IV. *If $f(x, \lambda)$ is continuous in (x, λ) and analytic in λ throughout the region (X, Λ) , the same will be true of the function*

$$F(x, \lambda) = \int_c^x f(a, \lambda) da,$$

c being any fixed point in X .

We prove first that F is analytic in λ . For this purpose it is sufficient to prove that it has a derivative with regard to λ at every point of (X, Λ) . We form the difference quotient

$$\frac{F(x, \lambda + \Delta\lambda) - F(x, \lambda)}{\Delta\lambda} = \int_c^x \frac{f(a, \lambda + \Delta\lambda) - f(a, \lambda)}{\Delta\lambda} da.$$

The integrand here is the continuous function D of Theorem I, which, as we there saw, approaches $\partial f / \partial \lambda$ uniformly as $\Delta\lambda$ approaches zero, provided we restrict ourselves to a closed sub-region of (X, Λ) , as we clearly have a right to do here. From a fundamental theorem on uniform convergence it follows that the right hand side of the last written formula approaches a limit, namely

$$\int_c^x \frac{\partial f(a, \lambda)}{\partial \lambda} da.$$

Thus the analytic character of F in λ is established.

Consider now the difference

$$\begin{aligned} F(x' + \Delta x, \lambda' + \Delta\lambda) - F(x', \lambda') &= [F(x' + \Delta x, \lambda' + \Delta\lambda) - F(x', \lambda' + \Delta\lambda)] \\ &\quad + [F(x', \lambda' + \Delta\lambda) - F(x', \lambda')] \end{aligned}$$

where (x', λ') is an arbitrary point of (X, Λ) and $\Delta x, \Delta\lambda$ are so restricted that $(x' + \Delta x, \lambda' + \Delta\lambda)$ also lies in (X, Λ) .

Since $F(x, \lambda)$ is analytic, and therefore continuous, in λ at the point (x', λ') , it follows that, however small ϵ may be, a positive constant δ_1 can be so chosen that when $|\Delta\lambda| < \delta_1$ the second parenthesis on the right of the last written formula is in absolute value less than $\frac{1}{2}\epsilon$. The first parenthesis may be written.

$$\int_{x'}^{x' + \Delta x} f(a, \lambda' + \Delta\lambda) da$$

and it is readily seen that a constant δ_2 can be so chosen that when $|\Delta x| < \delta_2$ this integral is in absolute value less than $\frac{1}{2}\epsilon$. If we denote the smaller of the two constants δ_1, δ_2 by δ , it follows that when $|\Delta x| < \delta, |\Delta\lambda| < \delta$

$$|F(x' + \Delta x, \lambda' + \Delta \lambda) - F(x', \lambda')| < \epsilon,$$

and thus the continuity of F at the point (x', λ') is established. The proof just given establishes also the

COROLLARY. *The derivative with regard to λ of F may be found by differentiating under the integral sign.*

THEOREM V. *If we develop by Taylor's Theorem about a point λ_0 of Λ a function $f(x, \lambda)$ which is continuous in (x, λ) and analytic in λ throughout (X, Λ) , and if we denote by Λ' a circle, inclusive of its circumference, which lies wholly in Λ and whose centre is at λ_0 , then the coefficients of this development are functions of x which are continuous in X , and the development converges uniformly in (x, λ) throughout the region (X', Λ') where X' is any closed sub-region of X .*

Let the development here referred to be

$$(1) \quad f(x, \lambda) = f_0(x) + f_1(x)(\lambda - \lambda_0) + f_2(x)(\lambda - \lambda_0)^2 + \dots$$

That the coefficients are continuous functions of x is an immediate consequence of the corollary to Theorem I, since $f_n(x)$ differs only by a constant factor from the value of $\partial^n f / \partial \lambda^n$ at the point λ_0 .

In order to establish the uniform convergence of (1) we proceed as follows.* Let ρ be the radius of Λ' . With λ_0 as centre construct a second circle whose circumference lies wholly in Λ but whose radius r is greater than ρ . The points on the circumference of this circle form a closed region which we will call T . Let M be the maximum of $|f(x, \lambda)|$ in the closed region (X', T) . Then by a well-known theorem on power-series (Cf. Forsyth, Theory of Functions, 2nd Ed. p. 34)

$$M \geq |f_n(x)| r^n.$$

Consequently throughout (X', Λ') the terms of (1) do not exceed in absolute value the corresponding terms of the series

$$M + M \frac{\rho}{r} + M \left(\frac{\rho}{r} \right)^2 + \dots$$

Since this is a convergent series of positive constant terms, the uniform convergence of (1) in (X', Λ') is established.

* For a more general formulation of this part of the theorem Cf. the Lemma on page 40 of my *Introduction to the Study of Integral Equations*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 10.

We have at the same time established the

COROLLARY. *Not only (1) but also the series*

$$|f_0(x)| + |f_1(x)| |\lambda - \lambda_0| + |f_2(x)| |\lambda - \lambda_0|^2 + \dots$$

is uniformly convergent in (x, λ) throughout the region (X', Λ') .

THEOREM VI. *If $f(x, \lambda)$ is continuous in (x, λ) and analytic in λ throughout the region (X, Λ) and possesses a derivative $\partial f / \partial x$ continuous in (x, λ) throughout this region, and if we develop $f(x, \lambda)$ by Taylor's Theorem about a point λ_0 of Λ , the coefficients of this series have derivatives continuous throughout X , and, if we differentiate the series term by term with regard to x , we obtain the Taylor's development of $\partial f / \partial x$ about the point λ_0 .**

If we write the Taylor's Series, as above, in the form (1), we have by a well-known formula of Cauchy

$$f_n(x) = \frac{1}{2\pi i} \int \frac{f(x, t)}{(t - \lambda_0)^{n+1}} dt$$

where the integral may be extended in the positive direction about a circumference C lying in Λ and having λ_0 as centre. We form the difference-quotient

$$\frac{f_n(x + \Delta x) - f_n(x)}{\Delta x} = \frac{1}{2\pi i} \int \frac{f(x + \Delta x, t) - f(x, t)}{\Delta x} \frac{dt}{(t - \lambda_0)^{n+1}}.$$

A reference to Lemma 2 shows that, as Δx approaches zero, the integrand converges uniformly in t for all points on C to the limit

$$\frac{\partial f(x, t)}{\partial x} \cdot \frac{1}{(t - \lambda_0)^{n+1}}.$$

Consequently $f_n(x)$ has a derivative given by the formula

$$f'_n(x) = \frac{1}{2\pi i} \int \frac{\partial f(x, t)}{\partial x} \cdot \frac{dt}{(t - \lambda_0)^{n+1}}.$$

This is, however, precisely the formula for the coefficient of $(\lambda - \lambda_0)^n$ in the Taylor's development of $\partial f(x, \lambda) / \partial x$ about the point λ_0 . Consequently that part of our theorem which says that this last development may be obtained by differentiating the series (1) term by term with regard to x is established. The fact that $f'_n(x)$ is continuous throughout X now follows at once by a ref-

* That $\partial f(x, \lambda) / \partial x$ is analytic in λ and therefore admits of a Taylor's development, we see from Theorem III.

erence to Theorem V, since that theorem may be applied to the development of $\partial f(x, \lambda)/\partial x$.

THEOREM VII. *If we denote by Λ a circle, exclusive of its circumference, whose centre is λ_0 and whose radius is R , and if the functions $f_0(x), f_1(x), \dots$ are continuous in X and a constant M exists such that throughout X*

$$|f_n(x)| R^n < M \quad (n = 0, 1, 2, \dots),$$

then throughout the region (X, Λ) the series

$$(1) \quad f_0(x) + f_1(x)(\lambda - \lambda_0) + f_2(x)(\lambda - \lambda_0)^2 + \dots$$

represents a function continuous in (x, λ) and analytic in λ .

For let Λ' be a circle, *inclusive* of its circumference, with centre at λ_0 and radius $r < R$. For all points of Λ' we have

$$|f_n(x)| |\lambda - \lambda_0|^n \leq |f_n(x)| r^n < M \left(\frac{r}{R}\right)^n.$$

Consequently, for all such points, the terms of (1) are in absolute value less than the corresponding terms of the series

$$M + M \frac{r}{R} + M \left(\frac{r}{R}\right)^2 + \dots$$

and, since this is a convergent series of positive constant terms, it follows that (1) is uniformly convergent in (x, λ) throughout the region (X, Λ') . Accordingly (1) represents a function of (x, λ) continuous throughout this region, and consequently, since Λ' can be made to include any arbitrarily chosen point of Λ , the function represented by (1) is continuous in (x, λ) throughout (X, Λ) . That it is analytic in λ throughout this region is obvious from the fact that (1) is a power-series in $\lambda - \lambda_0$.

The above theorem remains true if by Λ is understood the same circle as above *including* its circumference, and the condition referring to the inequality in Theorem VII is replaced by the condition that the series

$$|f_0(x)| + |f_1(x)| R + |f_2(x)| R^2 + \dots$$

be uniformly convergent in X . Here series (1) can be seen to be uniformly convergent in (X, Λ) . This modification of Theorem VII is less general than that theorem except in the one particular that the circumference of Λ is included.